



Differential Geometry II - Smooth Manifolds

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Lecturer: Dr. N. Tsakanikas

Assistant: L. E. Rösler

Exercise Sheet 11 – Solutions

Exercise 1: Let V be a smooth vector field on a smooth manifold M , let $J \subseteq \mathbb{R}$ be an interval, and let $\gamma: J \rightarrow M$ be an integral curve of V . Prove the following assertions:

(a) *Rescaling lemma:* For any $a \in \mathbb{R}$, the curve

$$\tilde{\gamma}: \tilde{J} \rightarrow M, t \mapsto \gamma(at)$$

is an integral curve of the vector field $\tilde{V} := aV$ on M , where $\tilde{J} := \{t \in \mathbb{R} \mid at \in J\}$.

(b) *Translation lemma:* For any $b \in \mathbb{R}$, the curve

$$\hat{\gamma}: \hat{J} \rightarrow M, t \mapsto \gamma(t+b)$$

is also an integral curve of V on M , where $\hat{J} := \{t \in \mathbb{R} \mid t+b \in J\}$.

Solution:

(a) If $t \in \tilde{J}$, then

$$\tilde{\gamma}'(t) = a\gamma'(at) = aV_{\gamma(at)} = \tilde{V}_{\tilde{\gamma}(t)}.$$

(b) If $t \in \hat{J}$, then

$$\hat{\gamma}'(t) = \gamma'(t+b) = V_{\gamma(t+b)} = V_{\hat{\gamma}(t)}.$$

Exercise 2 (to be submitted):

(a) Compute the integral curves of the Euler vector field V on \mathbb{R}^n .

(b) Consider the smooth manifold $M \subset \mathbb{R}^2$ and the smooth vector field $Y \in \mathfrak{X}(M)$ from [Exercise Sheet 10, Exercise 2(d)]. Find the maximal integral curve of Y starting at the point $(1, 1) \in M$ and describe its image geometrically.

(c) Compute the flow of each of the following smooth vector fields on \mathbb{R}^2 :

(i) $U = y \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

$$(ii) \quad V = x \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial y}.$$

$$(iii) \quad W = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}.$$

(d) Consider the smooth vector field $X_2 \in \mathfrak{X}(\mathbb{R}^3)$ from [Exercise Sheet 10, Exercise 2(c)]. Compute its flow and examine whether it is a complete vector field on \mathbb{R}^3 .

Solution:

(a) Recall that the value of the Euler vector field V at a point $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ is

$$V_x = x^1 \frac{\partial}{\partial x_1} \Big|_x + \dots + x^n \frac{\partial}{\partial x_n} \Big|_x.$$

If $\gamma: J \rightarrow \mathbb{R}^n$ is a smooth curve, written in standard coordinates as

$$\gamma(t) = (\gamma^1(t), \dots, \gamma^n(t)),$$

then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V translates to

$$\dot{\gamma}^j(t) = \gamma^j(t) \quad \text{for every } 1 \leq j \leq n,$$

which yields

$$\gamma^j: J = \mathbb{R} \rightarrow \mathbb{R}, \quad \gamma^j(t) = c_j e^t, \quad 1 \leq j \leq n,$$

for some constants $c_j \in \mathbb{R}$. Therefore, the unique maximal integral curve of V starting at $p = (p^1, \dots, p^n) \in \mathbb{R}^n$ is the smooth curve

$$\gamma: \mathbb{R} \rightarrow \mathbb{R}^n, \quad t \mapsto (p^1 e^t, \dots, p^n e^t).$$

In particular, it is the constant curve $t \in \mathbb{R} \mapsto (0, \dots, 0) \in \mathbb{R}^n$ for $p = (0, \dots, 0)$, while it is a smooth immersion for $p \in \mathbb{R}^n \setminus \{0\}$; see Exercise 5.

Observation: The Euler vector field V is a *complete* vector field on \mathbb{R}^n .

(b) If $\gamma: J \rightarrow \mathbb{R}^2$ is a smooth curve in M , written in coordinates as $\gamma(t) = (\gamma^1(t), \gamma^2(t))$, then the condition $\gamma'(t) = Y_{\gamma(t)}$ for γ to be an integral curve of Y translates to the system

$$\frac{d\gamma^1}{dt}(t) = \gamma^2(t) \quad \text{and} \quad \frac{d\gamma^2}{dt}(t) = 0 \quad \text{for } t \in J.$$

Hence, γ^2 is a constant function and γ^1 is an affine function. Since we also require that $\gamma(0) = (1, 1)$, we infer that the unique maximal integral curve of Y starting at $(1, 1) \in M$ is the smooth curve

$$\gamma: J = (-1, +\infty) \rightarrow M \subseteq \mathbb{R}^2, \quad t \mapsto (t + 1, 1)$$

whose image is the straight line segment $\{(x, 1) \in \mathbb{R}^2 \mid x > 0\}$.

(c) We only provide the details for the solution of (ii), since the procedure for all other cases is essentially identical.

(c)(i) The unique maximal integral curve of U starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$, $t \mapsto (\frac{1}{2}t^2 + p^2t + p^1, t + p^2)$, which is a smooth immersion; see *Exercise 5(b)*. Hence, the flow of the complete vector field U is the map

$$\theta_U: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, (t, (x, y)) \mapsto \left(\frac{1}{2}t^2 + yt + x, t + y \right).$$

(c)(ii) Observe first that the unique maximal integral curve of V starting at $p = (0, 0)$ is the constant curve $\gamma_0: \mathbb{R} \rightarrow \mathbb{R}^2$, $t \mapsto (0, 0)$; see *Exercise 5(a)*. Now, if $\gamma: J \rightarrow \mathbb{R}^2$ is a smooth curve, written in standard coordinates as $\gamma(t) = (\gamma^1(t), \gamma^2(t))$, then the condition $\gamma'(t) = V_{\gamma(t)}$ for γ to be an integral curve of V translates to

$$\begin{aligned} \dot{\gamma}^1(t) &= \gamma^1(t), \\ \dot{\gamma}^2(t) &= 2\gamma^2(t). \end{aligned}$$

Therefore, there are constants $c_1, c_2 \in \mathbb{R}$ such that

$$\begin{aligned} \gamma^1: J = \mathbb{R} \rightarrow \mathbb{R}, \quad \gamma^1(t) &= c_1 e^t, \\ \gamma^2: J = \mathbb{R} \rightarrow \mathbb{R}, \quad \gamma^2(t) &= c_2 e^{2t}, \end{aligned}$$

so the unique maximal integral curve of V starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$, $t \mapsto (p^1 e^t, p^2 e^{2t})$, which is a smooth immersion for $p \in \mathbb{R}^2 \setminus \{(0, 0)\}$; see *Exercise 5(b)*.

In conclusion, V is a complete vector field on \mathbb{R}^2 whose flow is the map

$$\theta_V: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, (t, (x, y)) \mapsto (x e^t, y e^{2t}).$$

(c)(iii) The unique maximal integral curve of W starting at $p = (p^1, p^2) \in \mathbb{R}^2$ is the smooth curve $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$, $t \mapsto (p^1 e^t, p^2 e^{-t})$, which is a smooth immersion for $p \in \mathbb{R}^2 \setminus \{(0, 0)\}$; see *Exercise 5(b)*. Hence, the flow of the complete vector field W is the map

$$\theta_W: \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, (t, (x, y)) \mapsto (x e^t, y e^{-t}).$$

(d) To compute the integral curves of

$$X_2 = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^3),$$

we have to solve the following system of ODEs:

$$\begin{cases} \dot{\gamma}^1(t) &= -\gamma^2(t), \\ \dot{\gamma}^2(t) &= \gamma^1(t), \\ \dot{\gamma}^3(t) &= 0. \end{cases}$$

Combining the first two equations we obtain $\ddot{\gamma}^1 = -\gamma^1$, and thus

$$\gamma^1(t) = a \cos(t) - b \sin(t), \quad t \in \mathbb{R}$$

for some $a, b \in \mathbb{R}$. Using the first equation above, this then also gives

$$\gamma^2(t) = a \sin(t) + b \cos(t), \quad t \in \mathbb{R}.$$

On the other hand, the third equation above gives $\gamma^3(t) \equiv c$, $t \in \mathbb{R}$, for some $c \in \mathbb{R}$. One then verifies that the curve

$$\begin{aligned} \gamma: \mathbb{R} &\rightarrow \mathbb{R}^3 \\ t &\mapsto (a \cos(t) - b \sin(t), a \sin(t) + b \cos(t), c) \end{aligned}$$

is indeed a solution to the above system of equations. Notice that $\gamma(0) = (a, b, c)$, so the parameters are uniquely determined by the starting value.

In conclusion, the maximal flow domain of X_2 is $\mathcal{D} = \mathbb{R} \times \mathbb{R}^3$ and the maximal flow of X_2 is given by

$$\begin{aligned} \theta: \mathbb{R} \times \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ (t, (a, b, c)) &\mapsto (a \cos(t) - b \sin(t), a \sin(t) + b \cos(t), c). \end{aligned}$$

In particular, X_2 is complete.

Exercise 3: Let $\theta: \mathbb{R} \times M \rightarrow M$ be a smooth global flow on a smooth manifold M . Show that the infinitesimal generator V of θ is a smooth vector field on M , and that each curve $\theta^{(p)}: \mathbb{R} \rightarrow M$ is an integral curve of V .

Solution: By definition of the infinitesimal generator, we have

$$V_p = \left. \frac{d}{dt} \right|_{t=0} \theta(t, p) \quad \text{for all } p \in M. \quad (\star)$$

First, to show that V is smooth, we apply *Proposition 7.5*: Given an open subset U of M , a smooth real-valued function f on U , and a point $p \in U$, we have

$$\begin{aligned} Vf(p) &= V_p f = \left(\left. \frac{d}{dt} \right|_{t=0} \theta(t, p) \right) f \\ &= \left. \frac{d}{dt} \right|_{t=0} (f \circ \theta)(t, p) = \left. \frac{\partial}{\partial t} \right|_{(0, p)} (f \circ \theta)(t, p). \end{aligned}$$

Since the composite map $f \circ \theta$ is smooth, its partial derivative with respect to t is smooth as well. Thus, $Vf(p)$ depends smoothly on p , which implies that V is smooth.

Next, fix $p \in M$ and $s \in \mathbb{R}$. We have to show that

$$\left. \frac{d}{dt} \right|_{t=s} \theta(t, p) = V_{\theta(s, p)} \stackrel{(\star)}{=} \left. \frac{d}{dt} \right|_{t=0} \theta(t, \theta(s, p)).$$

By definition of a flow, we have

$$\theta(t + s, p) = \theta(t, \theta(s, p)),$$

and by first differentiating the above relation with respect to t and then evaluating at $t = 0$ we obtain the required identity.

Exercise 4:

- (a) *Naturality of integral curves:* Let $F: M \rightarrow N$ be a smooth map. Show that $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are F -related if and only if F takes integral curves of X to integral curves of Y .
- (b) *Naturality of flows:* Let $F: M \rightarrow N$ be a smooth map. Let $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$. Let θ be the flow of X and η be the flow of Y . Show that if X and Y are F -related, then for each $t \in \mathbb{R}$ it holds that $F(M_t) \subseteq N_t$ and $\eta_t \circ F = F \circ \theta_t$ on M_t :

$$\begin{array}{ccc} M_t & \xrightarrow{F} & N_t \\ \theta_t \downarrow & & \downarrow \eta_t \\ M_{-t} & \xrightarrow{F} & N_{-t} \end{array}$$

- (c) *Diffeomorphism invariance of flows:* Let $F: M \rightarrow N$ be a diffeomorphism. Show that if $X \in \mathfrak{X}(M)$ and if θ is the flow of X , then the flow of $F_*X \in \mathfrak{X}(N)$ is $\eta_t = F \circ \theta_t \circ F^{-1}$, with domain $N_t = F(M_t)$ for each $t \in \mathbb{R}$.

Solution:

- (a) Assume first that X and Y are F -related. Let γ be an integral curve of X . By definition and by *Proposition 3.18* we obtain

$$(F \circ \gamma)'(t) = dF_{\gamma(t)}(\gamma'(t)) = dF_{\gamma(t)}(X_{\gamma(t)}) = Y_{F(\gamma(t))} = Y_{(F \circ \gamma)(t)},$$

which shows that $F \circ \gamma$ is an integral curve of Y .

Assume now that F takes integral curves of X to integral curves of Y . Let $p \in M$ and let $\gamma: (-\varepsilon, \varepsilon) \rightarrow M$ be an integral curve of X starting at p . Then $\gamma(0) = p$ and $\gamma'(0) = X_p$. Moreover, by assumption, $F \circ \gamma: (-\varepsilon, \varepsilon) \rightarrow N$ is an integral curve of Y starting at $F(p)$, so $Y_{(F \circ \gamma)(0)} = (F \circ \gamma)'(0)$. Therefore, by *Proposition 3.18* we obtain

$$Y_{F(p)} = (F \circ \gamma)'(0) = dF_p(\gamma'(0)) = dF_p(X_p).$$

Since $p \in M$ was arbitrary, we conclude that X and Y are F -related.

- (b) Denote by \mathcal{D}_X (resp. \mathcal{D}_Y) the flow domain of θ (resp. η). Fix $t \in \mathbb{R}$ and let $p \in M_t$. Then $t \in \mathcal{D}_X^{(p)}$ and $\theta^{(p)}: \mathcal{D}_X^{(p)} \rightarrow M$ is the unique maximal integral curve of X starting at p . By part (a), $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F(p)$. Hence, by maximality, we obtain that $\mathcal{D}_X^{(p)} \subseteq \mathcal{D}_Y^{(F(p))}$, and thus $t \in \mathcal{D}_Y^{(F(p))}$, which shows that $F(p) \in N_t$. In conclusion, $F(M_t) \subseteq N_t$.

Finally, we have

$$F \circ \theta_t(p) = F(\theta(t, p)) \stackrel{(*)}{=} \eta(t, F(p)) = \eta_t \circ F(p),$$

where in $(*)$ we again used that $F \circ \theta^{(p)}$ is an integral curve of Y starting at $F(p)$ and thus it is equal to $\eta^{(F(p))}$ where it is defined (this uses the uniqueness part in the theorem about solutions to a system of ODEs).

- (c) Denote by η the flow of F_*X . Applying part (a) to both F and F^{-1} we infer that $F(M_t) \subseteq N_t$ and $F^{-1}(N_t) \subseteq M_t$, so that $F(M_t) = N_t$ for each $t \in \mathbb{R}$. Furthermore, the commutativity of the above diagram shows that $\eta_t = F \circ \theta_t \circ F^{-1}$ for all $t \in \mathbb{R}$.

Exercise 5: Let V be a smooth vector field on a smooth manifold M and let $\theta: \mathfrak{D} \rightarrow M$ be the flow generated by V . Prove the following assertions:

- (a) If $p \in M$ is a singular point of V , then $\mathfrak{D}^{(p)} = \mathbb{R}$ and $\theta^{(p)}$ is the constant curve $\theta^{(p)}(t) \equiv p$.
- (b) If $p \in M$ is a regular point of V , then $\theta^{(p)}: \mathfrak{D}^{(p)} \rightarrow M$ is a smooth immersion.

[Hint: Argue by contraposition and use the fundamental theorem on flows.]

Solution:

(a) If $V_p = 0$, then the constant curve $\gamma: \mathbb{R} \rightarrow M$, $t \mapsto p$ is clearly an integral curve of V , so it must be equal to $\theta^{(p)}$ by uniqueness and maximality.

(b) Assume that $\theta^{(p)}: \mathfrak{D}^{(p)} \rightarrow M$ is not a smooth immersion. Then $\theta^{(p)'}(s) = 0$ for some $s \in \mathfrak{D}^{(p)}$. Set $q := \theta^{(p)}(s)$ and note that $V_q = 0$, since $\theta^{(p)}$ is an integral curve of V . Thus, q is a singular point of V , and by part (a) we infer that $\mathfrak{D}^{(q)} = \mathbb{R}$ and that $\theta^{(q)}$ is the constant curve $\theta^{(q)}(t) \equiv q$. It follows from *Theorem 7.28(b)* that $\mathfrak{D}^{(p)} = \mathbb{R}$ as well, and for all $t \in \mathbb{R}$ the group law gives

$$\theta^{(p)}(t) = \theta_t(p) = \theta_{t-s}(\theta(s, p)) = \theta_{t-s}(q) = q.$$

For $t = 0$ we obtain $q = \theta^{(p)}(0) = p$, and hence $\theta^{(p)}(t) \equiv p$ and $V_p = \theta^{(p)'}(0) = 0$, which contradicts the assumption that p is a regular point of V . This finishes the proof of (b).

Remark. It can be shown that if V is a smooth vector field on a smooth manifold M and if $p \in M$ is a regular point of V , then there exist smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\frac{\partial}{\partial s^1}$. Therefore, a flow in a neighborhood of a regular point behaves, up to diffeomorphism, just like translation along parallel coordinate lines in \mathbb{R}^n .